Algebras of the SU(n) invariants: structure, representations and applications

1992 J. Phys. A: Math. Gen. 25 393

(http://iopscience.iop.org/0305-4470/25/2/019)

J. Phys. A: Math. Gen. **25** (1992) 393-399. Printed in the UK

Algebras of the SU(n) invariants: structure, representations and applications

V P Karassiov

Optics Department, Lebedev Physics Institute (FIAN), 53 Lenin Avenue, 117924 Moscow, USSR

Received 3 May 1991

Abstract. A new class of Lie-algebraic structures is determined within examination of bosonic oscillator systems with internal SU(n) symmetries. They are generated by the SU(n) vector invariants made up of bosonic operators and act complementarily to the SU(n) group on the Fock spaces. A full spectral analysis of the Fock spaces is given with respect to both SU(n) algebras and their complementary ones. Some physical applications of the results to composite models of many-body systems are also pointed out.

For many decades the symmetry approach has been widely and successfully used in the quantum theory of many-body systems (see e.g. [1-8]). Specifically, analysis of many-body problems within the second quantization method introduces in a natural way a symmetry formalism associated with oscillators of bosonic and fermionic types [1, 4-6].

Such an approach is especially fruitful in examining composite models with an internal symmetry since it allows us to display some hidden symmetries and other peculiarities of systems under consideration [6-9].

Indeed, let us consider many-body quantum oscillator systems which are associated with the creation and annihilation operators x_i^α and $\bar{x}_i^\alpha = (x_i^\alpha)^+$, respectively ($\alpha = 1, 2, \dots, n$; $i = 1, 2, \dots, m < \infty$, the superscript $+$ denotes the Hermitian conjugation). Here the superscript α labels 'internal' components of one-particle states that transform in accordance with the vector (fundamental) irreducible representation (irrep) D^1 (G) of a group G:

$$x_i^\alpha \overset{\mathrm{G}}{\to} u^{\alpha\beta} x_i^\beta \qquad \bar{x}_i^\alpha \overset{\mathrm{G}}{\to} (u^{\alpha\beta} x_i^\beta)^+ \qquad \|u^{\alpha\beta}\| \in D^1(\mathrm{G}) \tag{1}$$

where from hereon the summation is implied over repeated Greek superscripts. The operators $x_i^\alpha, \bar{x}_j^\beta$ satisfy the standard commutation relations (CR)

$$\begin{aligned} &[x_i^\alpha, x_j^\beta]_{\sigma(\lambda)} = x_i^\alpha x_j^\beta + \lambda x_j^\beta x_i^\alpha = 0 = [\bar{x}_i^\alpha, \bar{x}_j^\beta]_{\sigma(\lambda)} \\ &[\bar{x}_i^\alpha, x_j^\beta]_{\sigma(\lambda)} = \delta^{\alpha\beta}\delta_{ij} \qquad \sigma(\lambda) = \operatorname{sgn}\lambda \end{aligned} \tag{2}$$

where $\lambda = -1$ and 1 for bosonic and fermionic systems, respectively. The Hilbert spaces for these systems are the Fock spaces L_F spanned by the basic vectors

$$|\{n_i^\alpha\}\rangle = N(\{n_i^\alpha\}) \prod_{\{\alpha\}} (x_1^{\alpha_1})^{n_1^{\alpha_1}} (x_2^{\alpha_2})^{n_2^{\alpha_2}} \dots (x_m^{\alpha_m})^{n_3^{\alpha_3}} |0\rangle \tag{3}$$

where $|0\rangle$ is the vacuum vector: $\bar{x}_i^\alpha|0\rangle = 0$, $\forall\alpha$, i and N is a normalization constant. All physical operators including a Hamiltonian H are polynomials in variables x_i^α, $\bar{x}_j^\beta$, e.g.

$$H = \sum_{i,j} \omega_{ij}^{\alpha\beta} x_i^\alpha \bar{x}_j^\beta + \sum_i (c_i^\alpha x_i^\alpha + c_i^{\alpha*} \bar{x}_i^\alpha) + \text{higher powers} \tag{4}$$

where the asterisk $*$ denotes the complex conjugation.

Now we suppose that a Hamiltonian H is invariant with respect to the action (1) of the 'internal' symmetry group G. Then, according to the vector invariant theory [10], H depends polynomially only on some elementary G-invariants $I_r(\{x_i^\alpha, \bar{x}_j^\beta\})$ made up of G-vectors $x_i = (x_i^\alpha)$ and $\bar{x}_i = (\bar{x}_i^\alpha)$. Further, this G-invariance of H implies a possibility of picking out the G-invariant subspaces in L_F that one may interpret as the existence of kinematically coupled subsystems with G-invariant dynamics. In order to examine such composite subsystems within the general symmetry approach [3, 4] we need to construct C^*-algebras [11] of the G-invariant observables $k_m(\mathrm{G})$ and the G-invariant dynamic symmetry algebras $k_m^{(\lambda)}(\mathrm{G})$ in terms of $\{I_r(\{x_i^\alpha, \bar{x}_j^\beta\})\}$ as well as study representations of these algebras in the spaces L_F [9].

Efficient tools for solving these problems are the vector invariant theory [10] and the conception of complementary groups and algebras [8, 12]. Specifically, the complementarity theory allows us to decompose the space L_F into direct sum (with a simple spectrum)

$$L_F = \bigoplus_\alpha L_F^\alpha \tag{5}$$

where the subspaces L_F^α are irreducible with respect to the algebra $g \oplus k_m^{(\lambda)}(\mathrm{G})$ (g being the Lie algebra of G) and furthermore the label α determines simultaneously both an irrep $D^\alpha(g)$ of g and an dual irrep $D^\alpha(k_m^{(\lambda)}(\mathrm{G}))$ of $k_m^{(\lambda)}(\mathrm{G})$. From the physical point of view the decomposition (5) gives rise to some superselection rules [11] since the single spaces L_F^α with different α do not 'mix' under the time-evolution governed by a Hamiltonian $H \in k_m^{(\lambda)}(\mathrm{G})$. Thus the 'internal' symmetry algebra g 'induces' the 'hidden' dynamic symmetry algebra $k_m^{(\lambda)}(\mathrm{G})$.

This programme is simply and fruitfully realized in many-body physics for the groups $\mathrm{G} = 0(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ since in these cases the basic invariants $I_r(\{\ldots\})$ are bilinear combinations of the operators x_i^α and $\bar{x}_j^\beta$, and therefore algebras $k_m^{(\lambda)}(\mathrm{G})$ are well known finite-dimensional Lie algebras (see e.g. [12, 13]). However, for the groups $\mathrm{G} = \mathrm{SU}(n)$ and $\mathrm{SO}(n)$ the situation is more complicated. Specifically, for $n \geqslant 3$ the algebras $k_m^{(-1)}(\mathrm{SU}(n))$ and $k_m^{(-1)}(\mathrm{SO}(n))$ belong to new classes of infinite-dimensional Lie algebras [6, 7] associated with some deformations (cf [14]) of generalized oscillator algebras. The main aim of the present paper is to examine the situation in more detail for the case $\mathrm{G} = \mathrm{SU}(n)$, $D^1(\mathrm{G}) = D(1\dot{0}_{n-2})$, $\lambda = -1$, where $[p_1, \ldots, p_n]$ is the highest weight of the $\mathrm{SU}(n)$ irrep $D(p_1, \ldots, p_{n-1})$ and the dot as a superscript over 'a' in $\dot{a}_r$ means the repetition of a r times.

So, specialize our further analysis for bosonic systems ($\lambda = -1$). It is well known [10] that the set of the basic vector invariants $I_r(\{x_i, \bar{x}_j\})$ for the group $\mathrm{SU}(n)$ consists of the following constructions:

$$E_{ij} \equiv (x_i \bar{x}_j) = x_i^\alpha \bar{x}_j^\alpha = (E_{ji})^+ \qquad i, j = 1, \ldots, m \tag{6a}$$

$$X_{i_1 \ldots i_n} \equiv [x_{i_1} \ldots x_{i_n}] = \varepsilon^{\alpha_1 \ldots \alpha_n} x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n} \qquad i_1 < i_2 < \ldots < i_n \tag{6b}$$

$$\bar{X}_{i_1 \ldots i_n} \equiv [\bar{x}_{i_1} \ldots \bar{x}_{i_n}] = (X_{i_1 \ldots i_n})^+ \qquad \text{for } \lambda = -1 \tag{6c}$$

where $\varepsilon^{\cdots}$ is the invariant antisymmetric tensor. The entities (6) are generators of the C^*-algebra $k_m(SU(n)) \equiv k_m(n)$ of the *SU(n)-invariant conceptual observables* whose elements are formal power series in the variables (6) with their certain ordering.

Specifically, from the second Hilbert theorem of the vector invariant theory [10] for $\lambda = -1$ we have the identities ('syzygies')

$$X_{i_1\ldots i_n}X_{j_1\ldots j_n} - X_{j_1 i_2\ldots i_n}X_{i_1 j_2\ldots j_n} + \ldots + (-1)^n X_{j_1 i_1\ldots i_{n-1}}X_{i_n j_2\ldots j_n} = 0 \tag{7a}$$

$$X_{i_1\ldots i_n}E_{r_j} - X_{r i_2\ldots i_n}E_{i_1 j} + \ldots + (-1)^n X_{r i_1\ldots i_{n-1}}E_{i_n j} = 0 \tag{7b}$$

$$X_{i_1\ldots i_n}\bar{X}_{j_1\ldots j_n} = P_n(\{E_{ij}\}) \tag{7c}$$

and those obtained by the Hermitian conjugation of equations (7). Here $P_n(\{E_{ij}\})$ are polynomials of the nth order in variables $(6a)$ whose explicit form can be found from the algebraic identity [10] $\varepsilon^{\alpha_1\cdots\alpha_n}\varepsilon_{\beta_1\ldots\beta_n} = \det\|\delta^{\alpha_i}_{\beta_j}\|$. The identities (7) allow us to identify the algebras $k_m(n)$ as PI-algebras (algebras with polynomial identities) [6] on the Grassmann manifolds with the Plucker coordinates $X_{i_1\ldots i_n}$ [16].

Further, from the CRs (2) with $\lambda = -1$ we easily find CRs *for the quantities (6)*:

$$[E_{ij}, E_{rs}] \equiv [E_{ij}, E_{rs}]_- = \delta_{jr}E_{is} - \delta_{is}E_{rj} \tag{8a}$$

$$[X_{i_1\ldots i_n}, X_{j_1\ldots j_n}] = 0 = [\bar{X}_{i_1\ldots i_n}, \bar{X}_{j_1\ldots j_n}] \tag{8b}$$

$$[E_{rj}, X_{i_1\ldots i_n}] = \delta_{ji_1}X_{r i_2\ldots i_n} + \delta_{ji_2}X_{i_1 r i_3\ldots i_n} + \ldots \tag{8c}$$

$$[E_{rj}, \bar{X}_{i_1\ldots i_n}] = -(\delta_{ri_1}\bar{X}_{ji_2\ldots i_n} + \delta_{ri_2}\bar{X}_{i_1 j i_3\ldots i_3} + \ldots) \tag{8d}$$

$$[\bar{X}_{i_1\ldots i_n}, X_{j_1\ldots j_n}] = P'_n(\{E_{ij}\}) \tag{8e}$$

where $P'_n(\{E_{ij}\})$ are polynomials of the $(n-1)$th order in variables E_{ij} which are obtained by using the explicit form of the polynomials $P_n(\{E_{ij}\})$ in equation $(7c)$. Specifically, for the case $n = 2$ we have

$$[\bar{X}_{ij}, X_{kl}] = P'_2(\{E_{ij}\}) = -2(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}) - \delta_{il}E_{kj} + \delta_{ik}E_{lj} + \delta_{lj}E_{kl} - \delta_{jk}E_{il} \tag{9}$$

that allow us to close the CRs (8) and to introduce the so*(2*m*) Lie algebra structure on the set $I_m(2) \equiv \{X_{ij}, \bar{X}_{kl}, E_{ij}\}$ [6, 9].

It is not the case, however, for $n \geqslant 3$ because repeated CRs of $P'_n(\{E_{ij}\})$ with elements of the set $I_m(n) \equiv \{E_{ij}, X_{i_1\ldots i_n}, \bar{X}_{i_1\ldots i_n} \mid i = 1, \ldots, m\}$ contain elements of the $k_m(n)$ algebras with higher powers of $E_{ij}, X_{i_1\ldots i_m}, \bar{X}_{j_1\ldots j_m}$, and thus result in infinite-dimensional Lie algebras $k_n^{(-)}(n)$ [9]. But if we restrict ourselves by considering only the initial CRs (8) we obtain a new class of Lie-algebraic structures $I_m^{(-)}(n)$ which resemble deformations of usual Lie algebras (cf [14]). Indeed, the CRs $(8a)$–$(8d)$ are similar to those for elements of usual bosonic oscillator Lie algebras $u(m) \oplus h(m)$ [2–4], while the CR $(8e)$ represents a non-standard (non-parametric) polynomial deformation of the canonical CR. For example, in the case $n = m = 3$ we have

$$\begin{aligned}[\bar{X}_{123}, X_{123}] &= 6 + 9N + 3N^2 - \tfrac{1}{2}C_2(\mathrm{SU}(3)) \\ &= 3! + 3\sum_{i=1}^{3} E_{ii} + E_{11}E_{22} - E_{21}E_{12} + E_{22}E_{33} - E_{32}E_{23} \\ &\quad + E_{33}E_{11} - E_{13}E_{31}\end{aligned} \tag{10}$$

where $N = \frac{1}{3}\sum E_{ii}$, $C_2(\ldots)$ is the quadratic Casimir operator of the internal group SU(3).

We also note that with each algebra $I_m^{(-)}(n)$ one may associate a Lie algebra $k_n^{(*)}(n)$ if instead of the usual Lie bracket $[\cdot,\cdot]$ we use a new Lie bracket $[\cdot,\cdot]_* \equiv \mathrm{Pr}_{I_m}[\cdot,\,]$ where the symbol Pr_{I_m} stands for the projection on to Span $I_m(n)U\{cI\}$ (with I being the identity operator). In a sense the algebras $k_n^{(*)}(n)$ may be considered as special ('linearized') mutations [15] of the algebras $k_n^{(-)}(n)$ which can be used for a description of a generalized dynamics of the SU(n)-clusters.

Thus, the set (6) generates mutually related Lie-algebraic structures $k_n^{(-)}(n)$, $I_m^{(-)}(n)$ and $k_n^{(*)}(n)$ of three different types which are connected with oscillators on the Grassmann manifolds. Any of these algebras has two mutually conjugate finite-dimensional parabolic Lie subalgebras $b_+^{(m,n)} = \mathrm{Span}\{E_{ij}, X_{i_1\ldots i_n}\}$ and $b_-^{(m,n)} = \mathrm{Span}\{E_{ij}, \bar{X}_{i_1\ldots i_n}\}$ of the Grassmann oscillator algebras. In addition, the algebras $k_n^{(-)}(n)$ have a characteristic property of nilpotency

$$\begin{aligned} &ad_A^{n+1}B=0 \qquad A\in X_- = \mathrm{Span}\{X^-_{i\ldots i}\} \qquad B\in X_+=\mathrm{Span}\{X_{i\ldots i}\} \\ &ad_A B=[A,B] \qquad ad_A^n B = ad_A^{n-1}(ad_A B) \end{aligned} \tag{11}$$

which is useful for summing up the Baker–Campbell–Hausdorff series and developing a theory of generalized coherent states (GCS) [9].

For the physical applications we need to construct representations of the above-defined algebras in the spaces L_F. Below we outline a general scheme of the spectral analysis of the spaces L_F with respect to actions of the algebras su(n)$\oplus k_n^{(-)}(n)$ that also determine appropriate irreps of the algebras $I_m^{(-)}(n)$ and $k_n^{(*)}(n)$. For this aim we use the concept of complementary algebras and groups [12, 13].

We start from the simplest case, $n=2$, when we have $k_n^{(-)}(2)=k_n^{(*)}(2)$. As is known, the algebra $k_n^{(-)}(2)=\mathrm{so}^*(2m)$ acts complementarily to the algebra su(2)$\sim$sp(2) on the space L_F [9, 13] and the decomposition (5) takes the form

$$L_F = \bigoplus_{J\geq 0} L(J) \tag{12}$$

where the label J specifies both the SU(2) irrep $D(2J)$ and the appropriate so*($2m$) irrep $D^J(\mathrm{so}^*(2m))$ [9].

The subspaces $L(J)$ are spanned by the basic vectors $|J; M; \nu\rangle$ where the labels M and ν distinguish basic vectors within irrep $D(2J)$ and $D^J(\mathrm{so}^*(2m))$ respectively. The vectors $|J; M; \nu\rangle$ are linear combinations of the Fock states (3). In the papers [8, 17] a simple algorithm has been developed for explicitly constructing these vectors by using the techniques of generating invariants and GCS. We consider such constructions for the case $m=2$ which, however, elucidates the situation in the general case.

For $m=n=2$ the algebra $k_2^{(-)}(2)=\mathrm{so}^*(4)$ is decomposed into the direct sum $\mathrm{so}^*(4)=\mathrm{su}_{\mathrm{inv}}(2)\oplus\mathrm{su}(1,1)$ (with generators X_{12}, $\bar{X}_{12}$, $\frac{1}{2}(E_{11}+E_{22})+1$ and E_{12}, E_{21}, $\frac{1}{2}(E_{11}-E_{22})$ for the subalgebras su(1, 1) and su(2), respectively), and the basic vectors $|J; M; \nu\rangle$ have the form

$$\begin{aligned} |J;M;\nu\rangle &\equiv |J;M;\{T,t\}\rangle \\ &= N(J,M,T,t)(e_1\bar{u})^{J-M}(e_2\bar{u})^{J+M}[x_1u]^{J+t}[x_2u]^{J-t}(X_{12})^{T-J}|0\rangle \end{aligned} \tag{13}$$

where N is a normalization factor, $(e_i\bar{u})=e_i^\alpha\bar{u}^\alpha$, u and $\bar{u}$ are some intermediate boson operators, $e_i=(\delta_i^\alpha)$ are the reference vectors. The vectors (13) are generated by the action of the operators X_{12}^S on the $(2J+1)^2$-dimensional 'vacuum' subspace $L_\nu(J)=\mathrm{Span}\{|J;M;\{Jt\}\rangle;\ J=\text{constant}\}$ with the characteristic property

$$\bar{X}_{12}|\nu\rangle = 0 \qquad |\nu\rangle\in L_\nu(J). \tag{14}$$

In turn the space $L_\nu(J)$ is generated by means of the lowering operators $\Sigma_{i=1}^2 x_i^2\bar{x}_i^1 = E^{21}$ and E_{21} of two subalgebras su(2)$\subset$su(2)$\oplus$so*(4) on the highest vector $|J; J; \{JJ\}\rangle$.

Now we consider an action of the above algebra so*(4) on the vectors (13) using the CRs (8). We note that because of the definition of $k_2^{(-)}(2)=\text{so}^*(4)$ its action does not change the values of numbers J, M 'controlled' by the 'internal' algebra $\text{su}_{\text{int}}(2)$. Hence each space $L(J)$, $J\neq 0$, decomposes into the direct sum

$$L(J)=\bigoplus_M L(J,M)=\bigoplus_M \text{Span}\{|JM;\{Tt\}\rangle: J, M=\text{constant}\} \tag{15}$$

of the disjoint spaces $L(J,M)$ which are equivalent with respect to the action of the algebra $k_2^{(-)}(2)=\text{so}^*(4)$. Further, an action of the subalgebra $\text{su}(2)_{\text{inv}}\subset\text{so}^*(4)$ does not change the quantum number T while the operators X_{12} and $\bar{X}_{12}$ of the subalgebra $\text{su}(1,1)\subset\text{so}^*(4)$ raise and lower its value by one respectively. Thus each space $L(J,M)$ is a conjunction of the disjoint $\text{su}_{\text{inv}}(2)$-equivalent subspaces $L(J;M;T)=\text{Span}\{|J;M;\{Tt\}\rangle: J, M, T\geq J=\text{constant}\}$ which are 'intertwined' by the operators $X_{12}, \bar{X}_{12}$. Such an action of the algebra $k_2^{(-)}(2)=\text{so}^*(4)$ on the space $L(J,M)$ resembles that of usual oscillator algebra on the Fock space (cf [2, 4]) and allows us to obtain the space-carrier of the so*(4) irrep $D^J(\text{so}^*(4))$ starting from any vector of the 'vacuum space' $L_\nu(J)$. Similarly, one can show that all spaces $L(J,M)$ are the carrier-spaces of equivalent irreps of the algebra $I_2^{(-)}(2)$.

The above analysis provides a sample for realizing spectral analysis of the spaces L_F in the case of arbitrary m and n [9]. Therefore we outline its logical scheme and point out some peculiarities in the general case.

For arbitrary m and n the algorithm consists of determining 'vacuum spaces' $L_\nu(p_1\ldots p_{n-1})$ and then constructing their su(m)-equivalent replicas ($\text{su}(m)\subset k_n^{(-)}(n)$, $I_m^{(-)}(n)$, $k_n^{(-)}(n)$) by means action of operators $X_{i_1\ldots i_n}$ on the vectors $|v\rangle\in L_\nu(p_1,\ldots,p_{n-1})$. In turn the spaces $L_\nu(\ldots)$ are generated by means of lowering operators of the algebras $\text{su}_{\text{int}}(n)=\text{Span}\{E^{ij}\equiv\Sigma_{r=1}\, x_r^i\bar{x}_r^j,\ i\neq j,\ \tilde{E}^{ii}=E^{ii}-E^{i+1,i+1}\}$ and $\text{su}_{\text{inv}}(m)=\text{Span}\{E_{ij}, i\neq j, \tilde{E}_{ii}=E_{ii}-E_{i+1,i+1}\}\subset k_m^{(-)}(n)$ on the common highest vectors $|p_1\ldots p_{n-1};\max\rangle\equiv|\langle p_i\rangle\rangle$ satisfying the following equations:

$$\bar{X}_{i_1\ldots i_n}|\langle p_i\rangle\rangle=0 \tag{16a}$$

$$\tilde{E}^{ii}|\langle p_i\rangle\rangle=p_i|\langle p_i\rangle\rangle=\tilde{E}_{ii}|\langle p_i\rangle\rangle \qquad i=1,\ldots,n-1 \tag{16b}$$

$$E^{ij}|\langle p_i\rangle\rangle=0=E_{ij}|\langle p_i\rangle\rangle \qquad i<j \tag{16c}$$

whose general solutions have the form [8]

$$L\langle p_i\rangle\rangle=N\prod_{i=1}^{n-1}[x_1\ldots x_i e_{i+1}\ldots e_n]^{p_i}|0\rangle. \tag{17}$$

As a result we obtain at the final step of the algorithm the following specialization of equation (5):

$$L_F=\bigoplus_{\langle p_i\rangle} L(\langle p_i\rangle)=\bigoplus_{\langle p_i\rangle,\mu',\mu'',\gamma} L(\langle p\rangle;\mu';\{\mu'';\gamma\}) \tag{18}$$

where

$$L(\langle p_1\rangle;\mu';\{\mu'';\gamma\})=\text{Span}\{|\langle p_i\rangle;\mu';\{\mu'';\gamma\}\rangle\}$$

are carrier-spaces of the $\text{su}_{\text{int}}(n)$ irreps $D(\langle p_i\rangle)$ and of associated (dual to $D(\langle p_i\rangle)$) irreps of the algebras $k_m^{(-)}(n)$, $I_m^{(-)}(n)$ and $k_m^{(*)}(n)$; μ' and μ'' are the Gel'fand–Tsetlin patterns for the algebras $\text{su}_{\text{int}}(n)$ and $\text{su}_{\text{inv}}(m)$, respectively; γ is an extra label for distinguishing vectors within irreps of $k_n^{(-)}(n)$, etc. [8]. Basic vectors $|\langle p_i\rangle;\mu';\{\mu'';\gamma\}\rangle$ resemble in their appearance the structure of the vectors (13) but instead of monomials X_{12}^s we obtain some polynomials in variables $X_{i_1\ldots i_n}$.

A natural area of applications of the above results is in developing composite models with internal SU(n)-symmetries. Such models are governed by SU(n)-invariant Hamiltonian H_{inv} formulated in terms of elements of the algebras $k_m(n)$:

$$H_{\text{inv}} = cI + \sum_i \omega_i E_{ii} + \sum_{i,j} c_{ij} E_{ij} + \sum d_{i_1 \ldots i_n} X_{i_1 \ldots i_n} + \sum d^*_{i_1 \ldots i_n} \bar{X}_{i_1 \ldots i_n} + \text{higher powers}. \tag{19}$$

Specifically, some effective Hamiltonians in quantum polarization optics have this form [9].

The quantities $X_{i_1 \ldots i_n}$ and $\bar{X}_{i_1 \ldots i_n}$ may be interpreted as operators of creation and annihilation, respectively, of SU(n)-invariant clusters. But, unlike usual quantum particles (bosons and fermions) these clusters have unusual statistics, as follows from the CRs (8). In particular, in the case $n=2$ we obtain from (8) trilinear CRs

$$[\bar{X}_{rs}, [\bar{X}_{ij}, X_{kl}]] = (\delta_{jl}\delta_{rk} - \delta_{jk}\delta_{rl})\bar{X}_{is} + (\delta_{jl}\delta_{sk} - \delta_{jk}\delta_{sl})\bar{X}_{ri} + \ldots \tag{20}$$

which generalize Green's trilinear CR for para-fields and para-particles [2]. The CRs (8) also imply the general form of the number operator N_{cl} of such clusters [9]

$$N_{\text{cl}} = (1/n)\sum_i E_{ii} - C(\{E^{\alpha\beta}\}) = (1/n)\sum_i E_{ii} - \tilde{C}(\{E_{ij}\}) \tag{21}$$

where $C(\ldots)$ are some SU(n)-invariant nonlinear functions of the SU(n) generators $E^{\alpha\beta}$ which are multiple to the identity operator I on each subspace $L(\langle p_i \rangle)$ from (18). Specifically, for $m=n=2$ we have

$$C(\{E^{\alpha\beta}\}) = -\tfrac{1}{2} + \tfrac{1}{2}(1 + 2(E^{12}E^{21} + E^{21}E^{12}) + (E^{11} - E^{22})^2)^{1/2}. \tag{22}$$

Thus, also taking into account (7), we see that internal SU(n)-symmetry yields us a scheme of a generalized paraquantization with constraints (cf [18]) on the spaces $L_{\text{F}} = \bigoplus L(\langle p_i \rangle)$. Because of non-trivial dimensions of the 'vacuum subspaces' $L_\nu(\langle p_i \rangle)$ we can develop models with spontaneously broken and hidden symmetries within the above formalism.

Another interesting line of investigation here is in examining the possibility of constructing canonical bases of observables Y_a, $\bar{Y}_b$ ($[\bar{Y}_b, Y_a] = \delta_{ab}$) in terms of elements of algebras $k_m(n)$. This way seems to be promising since, following the general scheme [19], we obtained in [9] explicit expressions for Y, $\bar{Y}$ in the case $m=n$:

$$Y = \sum_{J \geqslant 0} C_J(\langle p_i \rangle)(X_{12\ldots n})^{J+1}(\bar{X}_{12\ldots n})^J \qquad \bar{Y} = (Y)^+ \tag{23}$$

where the coefficients C_r are determined from a set of recurrence relations depending on signatures $\langle p_i \rangle$ of subspaces $L(\langle p_i \rangle)$. Specifically, on the SU(n)-scalar subspaces $L(\langle \dot{O} \rangle)$ coefficients $C_r = C_r(\langle \dot{0} \rangle)$ satisfy the relations

$$\begin{gathered} \sum C_r l^{(r)}(l+1)^{(r)} \ldots (l+n-1)^{(r)} = [(l+n)^{(n-1)}]^{-1/2} \qquad l = 0, 1, \ldots \\ l^{(r)} = l!/(l-r)! \end{gathered} \tag{24}$$

From equation (24) we easily find the rational generating function for C_r:

$$\begin{gathered} \sum_{r=0}^{\infty} (-t)^r C_r = \left[{}_0F_{n-1}\left(\begin{matrix} 0 \\ \{2,3,\ldots,n\} \end{matrix} ; -t \right) \right]^{-1} \\ \sum_{l=0}^{\infty} (-t)^l [l^{(l)}(l+1)^{(l)} \ldots (l+n-1)^{(l)}]^{-1} [(l+n-1)^{(n-1)}]^{-1/2} \end{gathered} \tag{25}$$

where ${}_pF_q(\ldots)$ is the generalized hypergeometric series [20].

Such developments can be useful in analysing composite models of many-body quantum systems of arbitrary physical nature (photons, phonons, etc). Some examples of solving certain problems in polarization quantum optics have been considered within this approach in [9].

The results obtained provide a mathematical tool for analysing composite models with internal SU(n)-symmetry only at algebraic level. However, for examining time evolution governed by Hamiltonians (19) we need to develop group-theoretical aspects of the theory, in particular, generalized coherent states of algebras $k_n^{(-)}(n)$, etc. Without considering this problem in detail we note here that the 'Glauber' GCS $\exp(\alpha Y - \alpha^* \bar{Y})|0\rangle$ are well determined by power series, while it is not the case for GCS $\exp(\zeta X_{1\ldots n} - \zeta^* \bar{X}_{1\ldots n})|0\rangle$ (cf [21]).

It is also of interest to extend our analysis by common consideration of both internal and the spacetime Poincaré symmetries. The 'Grassmann nature' of the SU(n)-clusters $X_{i_1\ldots i_n}$ gives hope that we can obtain along this line certain results which are useful for developments in string theory (cf [16]) and for analysing nonlinear phenomena and coherent structures in strongly interacting many-body systems. Finally we note that formal aspects of the above analysis may be completely extended for the case $G = SO(n)$. Another generalization is obtained by involving considerations other than D^1 (G) irrep of 'internal' groups G. Specifically, in SU(n)-invariant field theories one must consider both fundamental and conjugate irrep (cf [12, 18]).

References

[1] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover)

[2] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Tokyo University Press)

[3] Malkin I A and Man'ko V I 1979 *Dynamics Symmetries and Coherent States of Quantum Systems* (Moscow: Nauka)

[4] Perelomov A M 1987 *Generalized Coherent States and Their Applications* (Moscow: Nauka)

[5] Jordan P 1935 *Z. Phys.* **94** 531

[6] Karassiov V P 1989 *Topological Phases in Quantum Theory* (Singapore: World Scientific) p 400

[7] Karassiov V P 1988 *Sov. Phys.-P.N.Lebedev Phys. Inst. Reports* no 9 3

[8] Karassiov V P 1988 *Group-theoretical Methods in Fundamental and Applied Physics* (Moscow: Nauka) p 54

[9] Karassiov V P 1990 *Preprint* Lebedev Physics Institute (FIAN), Moscow, no 137; 1991 *J. Sov. Laser Res.* **12** 147

[10] Weyl H 1939 *The Classical Groups* (Princeton, NJ: Princeton University Press)

[11] Barut A O and Racka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN—Polish Sci. Publ.)

[12] Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772

[13] Alisauskas S I 1983 *Sov. J. Part. Nucl.* **14** 563

[14] Sklyanin E K 1982 *Funkt. Anal. Pril.* **16** 27, 263

[15] Myung H C and Sagle A 1987 *Hadronic J.* **10** 35

[16] Pressley A and Segal G 1986 *Loop Groups* (Oxford: Clarendon)

[17] Karassiov V P 1987 *J. Phys. A: Math. Gen.* **20** 5061

[18] Dirac P A M 1967 *Lectures on Quantum Field Theory* (New York: Yeshiva University Press)

[19] Brandt R A and Greenberg O W 1969 *J. Math. Phys.* **10** 1168

[20] Bateman H and Erdelyi A 1959 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)

[21] Fisher R A, Nieto M M and Sandberg V D 1984 *Phys. Rev.* D **29** 1107